THE RESIDUAL POWER SERIES ALGORITHM FOR SOLVING VARIABLE-DEPTH SHALLOW WATER EQUATIONS

Emad A. Az-Zo'bi
Department of Mathematics and Statistics, Faculty of Science, Mutah University, Jordan
E-mail: eaaz2006@yahoo.com, eaaz2006@mutah.edu.jo

ABSTRACT. *In this paper, the generalized residual power series technique (RPSM) is employed to obtain analytic-numeric solution for the famous hyperbolic couple of conservation laws, known by the variable-depth shallow water equations. The recently developed Algorithm is based on combining the Taylor series solutions and the well-defined residual functions. Convergence and error estimations for considered truncated series solutions are discussed briefly. Without need for unrealistic assumptions, the method is successfully applied for constructing approximate solutions of high accuracy for considered problem. The obtained solutions and corresponding absolute errors are calculated and shown graphically with the aid of Mathematica software package. The RPSM is simple, applicable, and reduces the size of computations.*

Keywords: Residual power series method, Shallow water equations, Conservation laws, Numerical solution.

1. INTRODUCTION

The notion of conservation laws is fundamental to understand many physical world phenomena arise in fluid mechanics, solid state physics, plasma physics, plasma waves and chemical physics. Also, conservation laws are fundamental laws in other fields of science such as engineering, biology, chemistry and geology. While most of such problems are modeled by differential equations, the development of constructing exact, semi-analytic, and numerical solutions of differential equations plays a vital role to allow physicists drawing conclusions in an efficient way.

The (1+1)-dimensional hyperbolic system of conservation laws, known by the shallow water equations, and governed by the couple of partial differential equations [1-2]

$$u_t + (uv)_x = 0, v_t + \left(u + 0.5v^2\right)_x = H'(x). \qquad (1)$$

models the flow of water in an infinitely wide rectangular cross-section, frictionless and smoothly varying bottom surface channel. In Eq.(1), $u(x,t)$ denotes the total height above the bottom of the channel, $v(x,t)$ represents the fluid velocity, and the analytic function $H(x)$ is the depth of a point from a fixed reference level of the water. The two independent variables x and t are the distance along the direction of flow and the time, respectively.

When the bottom is not flat, no exact analytical solution for the shallow water equations Eq.(1) could be found [2]. Authors in [3] proved the convergence of a weak solution for non-strictly hyperbolic system using the theory of compensated compactness. In order to construct approximate solutions of hyperbolic and hyperbolic-elliptic systems of conservation laws, several numeric-analytic schemes were applied. Among these attempts, we list the finite difference method [4], the finite element method [5], the Adomian decomposition method and its variants [6-11], the variational iteration method [12-13], differential and reduced differential transform method [14-18], the simple equation method [19], the perturbation iteration algorithm and sine-Gordon expansion method [20].

The motivation of the current work is to handle the shallow water system Eq.(1) by applying the residual power series method (shortly RPSM). To treat fuzzy differential equations, The RPSM was first employed by Abu Arqub [21]. Later, many authors have been applied this scheme for processing differential, integral, and integro-differential equations and systems of integer and fractional orders. For example, the generalized Burger-Huxley equation [22], isentropic flow of an inviscid gas model [23], higher dimensional telegraph equation [24], van der Waals p-system [25], nonlinear diffusion equation [26], one-dimensional shock wave equation [27] and fractional Zakharov-Kuznetsov equation [28]. See also the references therein.

2. MATERIAL AND METHODS

In this section, the procedure of residual power series scheme is described. For this purpose, consider the (1+1)-dimensional system of conservation laws in general form

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U};x,t) = \mathbf{f}(x,t), \ \mathbf{U}(x,t_0) = \mathbf{U}_0(x). \qquad (2)$$

Where $\mathbf{U} = (u,v)$, $\mathbf{F} = (F,G)$ and $\mathbf{f} = (f,g)$ is the inhomogeneous vector-valued term which is assumed to be analytic on some domain containing t_0.

As in the Taylor series method, the solution in the RPSM case is assumed to be in a form of power series as follows

$$\mathbf{U}(x,t) = \mathbf{U}_0(x) + \sum_{k=1}^{\infty} \xi_k(x)(t-t_0)^k, \qquad (3)$$

where $\xi_k(x) = (\xi_1(x), \xi_2(x))$ is the sequence of function coefficients to be determined.

For natural m, the m^{th}-order approximate solution, which satisfies initial data, is defined by the m^{th}-Taylor polynomial

$$\mathbf{U}_m(x,t) = \mathbf{U}_0(x) + \sum_{k=1}^{m} \xi_k(x)(t-t_0)^k. \qquad (4)$$

One can easily obtain that the approximate solution satisfies the initial conditions. Consequently, starting with $m=1$ in Eq.(4), of $\mathbf{U} = (U_1, \mathrm{U}_2)$, the first approximation will be

$$\mathbf{U}_1(x,t) = \mathbf{U}_0(x) + \xi_1(x)(t-t_0),$$

subject to:

$$\lim_{t \to t_0} \mathbf{Res}_1(x,t) = 0,$$

where,

$$\mathbf{Res}_1(x,t) = D_t \mathbf{U}_1 + D_x \mathbf{F}(\mathbf{U}_1). \qquad (5)$$

is the well-defined analytic residual function. In the same manner, for $m = 2,3,\ldots$, the undetermined coefficients $\xi_m(x)$ can be obtained consecutively by solving the algebraic equation

$$\lim_{t \to t_0} \mathbf{Res}_m(x,t) = \lim_{t \to t_0}\left[D_t^m \mathbf{U}_m + D_t^{m-1} D_x \mathbf{F}(\mathbf{U}_m)\right] = \mathbf{0} \qquad (6)$$

Consecutively, the undetermined coefficients of power series solution in Eq.(3) can be obtained as follows:

$$\xi_k(x) = \frac{-1}{k!} D_t^{k-1}\left[D_x \mathbf{F}(\mathbf{U}_k)\right]_{t=t_0} \quad (7)$$

Convergence and error estimation for approximating the solution of the initial-value problem in Eq.(1) for the modified version of the RPSM have been studied and proved [22-26]. Main theorems are listed in what follow.

Theorem 2.1. The residual function $\mathbf{Res}_m(x,t)$ vanishes as m approaches the infinity.

Corollary 2.2. The truncated series solution given in Eq.(4) and obtained by the RPSM is the m^{th} - order Taylor expansion of $\mathbf{U}(x,t)$ about $t = t_0$.

As a result of these facts, the approximate series solution will be more and more close to the exact solution, in its Taylor series form, while m increases. Therefore, the Taylor's theorem tells that

$$\mathbf{U}(x,t) = \mathbf{U}_0 + \sum_{k=1}^{m} \xi_k (t-t_0)^k + \mathbf{R}_{\mathbf{U},m}(x,t),$$

where,

$$\mathbf{R}_{\mathbf{U},\mathrm{m}}(x,t) = \frac{1}{(m+1)!}\partial_\tau^{m+1}\mathbf{U}(x,\tau)(\tau-t_0)^{m+1},$$

is the reminder term with $t_0 \le \tau \le t < t_0 + R$, for some real number $R > 0$. Assume that $\mathbf{U}(x,t) \in C^{m+1}(D)\times C^{m+1}(D)$ is bounded. That is, positive constants C_1, C_2 exist with $\left|\partial_t^{m+1}U_1\right| < C_1$, and $\left|\partial_t^{m+1}U_2\right| < C_2$ on the indicated domain. For $C = C_1 + C_2$, an error bound for using the m^{th} - order truncated series solution is estimated to be

$$\left|\mathbf{U}(x,t) - \mathbf{U}_m(x,t)\right| < \frac{R^{m+1}C}{(m+1)!}. \quad (8)$$

To reduce the error bound while being something large for $R > 1$, i.e. to expand the domain of convergence, or to control this error within required tolerance, a multistep residual power series technique can be constructed and employed as in the case of Adomian-Rach decomposition method [29]. Respectively, the absolute and relative errors are defined normally, at each point $(x,t) \in D$, with known exact analytical solution as follows

$$\mathbf{E}_{abs}^{m}(\mathbf{U}(x,t)) = \left|\mathbf{U}(x,t) - \mathbf{U}_m(x,t)\right|. \quad (9)$$

and,

$$\mathbf{E}_{rel}^{m}(\mathbf{U}(x,t)) = \frac{\left|\mathbf{U}(x,t) - \mathbf{U}_m(x,t)\right|}{\left|\mathbf{U}(x,t)\right|}. \quad (10)$$

With unknown exact analytical solution, two formulas of absolute errors are defined by

$$\mathbf{E}_{abs}^{m}(\mathbf{U}(x,t)) = \left|\mathbf{U}_{m+1}(x,t) - \mathbf{U}_m(x,t)\right|. \quad (11)$$

or, by direct substitution into Eq.(3),

$$\mathbf{E}_{abs}^{m}(\mathbf{U}(x,t)) = \left|D_t\mathbf{U}_m + D_x\mathbf{F}(\mathbf{U}_m) - \mathbf{f}\right|. \quad (12)$$

In this case, the relative error is defined by

$$\mathbf{E}_{rel}^{m}(\mathbf{U}(x,t)) = \frac{\left|\mathbf{U}_{m+1}(x,t) - \mathbf{U}_m(x,t)\right|}{\left|\mathbf{U}_{m+1}(x,t)\right|}. \quad (13)$$

In other words, for the case of unknown exact solution, the approximate power solution of order m Eq.(4) is convergent to the exact solution $\mathbf{U}(x,t)$ if the sequence of nonnegative numbers $\{\alpha_m\}$, where

$$\alpha_m = \frac{\left|\mathbf{U}_{m+1}(x,t)\right|}{\left|\mathbf{U}_m(x,t)\right|}, \quad (14)$$

converges to 1 as m increases unboundedly. α_m denotes the m^{th} – rate of convergence. As $\alpha_m \to 1$, with sufficient small number of iterations m, faster convergence will be obtained.

3. RESULTS

Following the presented procedure discussed in Section 2, the truncated approximate height and velocity of water flow of order m are assumed to be

$$\begin{aligned} u_m(x,t) &= u_0(x) + \sum_{k=1}^{m} \xi_k(x)(t-t_0)^k \\ v_m(x,t) &= v_0(x) + \sum_{k=1}^{m} \eta_k(x)(t-t_0)^k \end{aligned}. \quad (15)$$

Sequentially, the pairs of unknown coefficients $(\xi_k(x), \eta_k(x))$, $k = 1,2,\ldots,m$, could be determined by solving the corresponding coupled algebraic equations

$$\frac{\partial_t^{k-1}}{k!}\begin{bmatrix} \partial_t u_k + \partial_x(u_k v_k) \\ \partial_t v_k + \partial_x\left(u_k + 0.5 v_k^2\right) - H'(x) \end{bmatrix}_{t=0} = \mathbf{0} \quad (16)$$

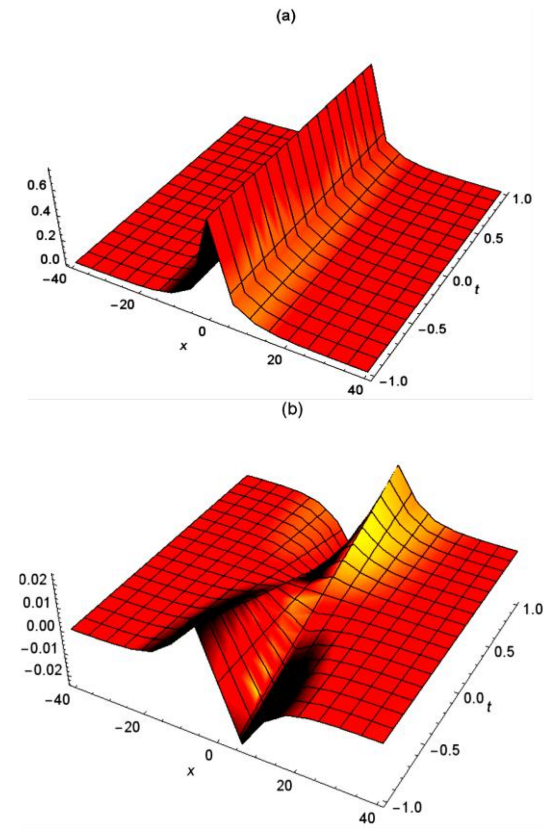


Fig. 1 The 3D behavior of approximate (a) height $u_5(x,t)$ and (b) velocity $v_5(x,t)$ of water on $D = \{|x| \le 40, |t| \le 1\}$.

The first few coefficients are listed below. Determining more coefficients, which implies more accuracy, is possible with aid of Mathematica software package.

$$\xi_1(x) = -v_0 u_0' - u_0 v_0'$$

$$\eta_1(x) = H' - u_0' - v_0 v_0'$$

$$\xi_2(x) = \frac{1}{2}\begin{pmatrix} -H' u_0' + u_0'^2 + 4v_0 u_0' v_0' + 2u_0 v_0'^2 \\ -u_0 H'' + u_0 u_0'' + v_0^2 u_0'' + 2u_0 v_0 v_0'' \end{pmatrix}$$

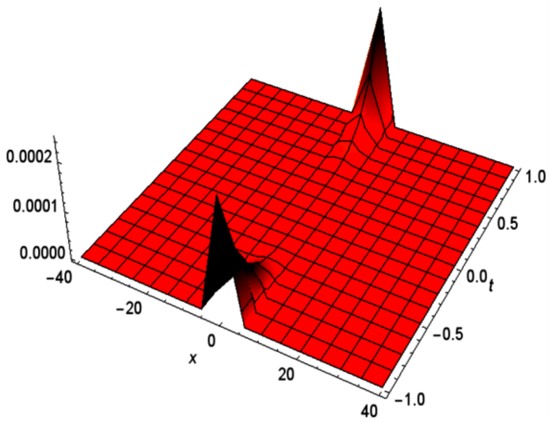
$$\eta_2(x) = \frac{1}{2}\begin{pmatrix} -H' v_0' + 3u_0' v_0' + 2v_0 v_0'^2 \\ -v_0 H'' + 2v_0 u_0'' + u_0 v_0'' + v_0^2 v_0'' \end{pmatrix}$$

$$\xi_3(x) = \frac{1}{6}\begin{pmatrix} 6H' u_0' v_0' - 8u_0'^2 v_0' - 18v_0 u_0' v_0'^2 \\ -6u_0 v_0'^3 + 5v_0 u_0' H'' + 5u_0 v_0'' H'' \\ +3v_0 H' u_0'' - 9v_0 u_0' u_0'' - 8u_0 v_0' u_0'' \\ -9v_0^2 v_0' u_0'' + 3u_0 H' v_0'' - 7u_0 u_0' v_0'' \\ -9v_0^2 u_0' v_0'' - 18u_0 v_0 v_0' v_0'' + 2u_0 v_0 H^{(3)} \\ -3u_0 v_0 u_0^{(3)} - v_0^3 u_0^{(3)} - u_0^2 v_0^{(3)} - 3u_0 v_0^2 v_0^{(3)} \end{pmatrix}$$

$$\eta_3(x) = \frac{1}{6}\begin{pmatrix} 3H' v_0'^2 - 11u_0' v_0'^2 - 6v_0 v_0'^3 - 2H' H'' \\ +4u_0' H'' + 5v_0 v_0' H'' + 3H' u_0'' - 5u_0' u_0'' \\ -15v_0 v_0' u_0'' + 3v_0 H' v_0'' - 12v_0 u_0' v_0'' \\ -7u_0 v_0' v_0'' - 9v_0^2 v_0' v_0'' + u_0 H^{(3)} + v_0^2 H^{(3)} \\ -u_0 u_0^{(3)} - 3v_0^2 u_0^{(3)} - 3u_0 v_0 v_0^{(3)} - v_0^3 v_0^{(3)} \end{pmatrix}$$

$$\vdots$$

(a)



(b)

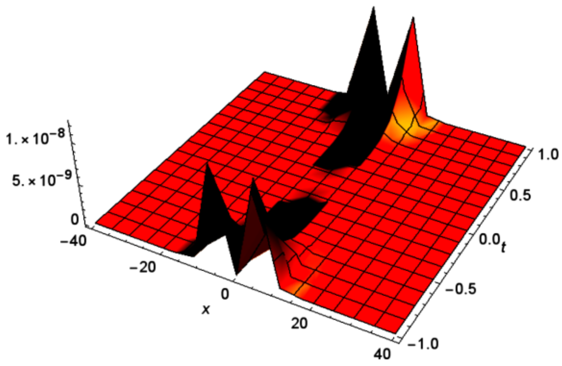


Fig. 2 Corresponding absolute errors for approximate (a) height $u_5(x,t)$ and (b) velocity $v_5(x,t)$ of water in Fig. 1 on D.

Numerically, the system of shallow water equations Eq.(1) subject to initial data [6-8]

$$H(x) = \frac{e^{-x^2}}{1+e^{-x^2}}, \qquad (17)$$

where

$$\left.\begin{array}{l} u(x,0) = H(x) + 0.25Sech(\beta x) \\ v(x,0) = 0 \end{array}\right., \quad 0 < \beta \le 1. \qquad (17)$$

is considered. The 5th-order approximate solutions are obtained and shown in Figure 1. With undetermined closed-form exact solutions, and for accuracy test purposes, the corresponding absolute errors given in Eq.(12) are exhibited in Figure 3. Comparing to other used methods, like the Adomian decomposition method [6], the variational iteration method [12], and the reduced differential transform method [15], our approach is more accurate. The highly accurate solutions make the obtained approximate solutions an acceptable as a criterion of comparison in next works. The convergence rates in Eq.(14) are listed for computed truncated series solutions to be

$\rho_0 = 1.16673$,

$\rho_1 = 1.00257$,

$\rho_2 = 1.00154$,

$\rho_3 = 1.00072$,

$\rho_4 = 1.00057$, …

Figure 3 shows the numerical behavior of water flow in the $u-v$ plane with step-size $h = 0.01$.

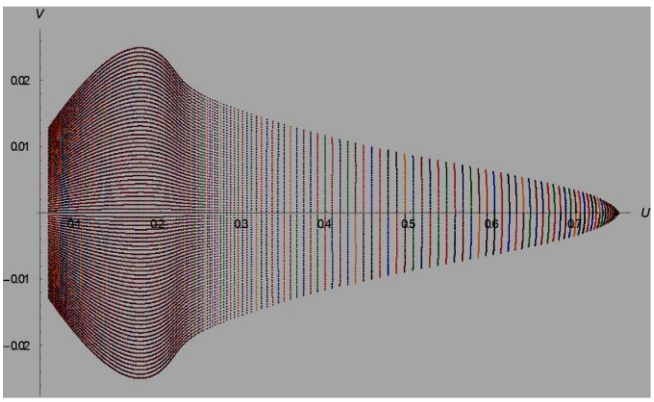


Fig. 3 Numerical plot of approximate solutions in the $u-v$ plane for $|x| \le 10$ and $|t| \le 1$.

4. DISCUSSION AND CONCLUSION

The recently presented semi-analytic residual power series method is successfully applied to the (1+1)-dimensional system of variable-depth shallow water equations. This system is well-known example of hyperbolic conservation laws. Our approach provides a solution in a form of Taylor's series expansion with coefficients determined consecutively in the mean of residual function. The method is applicable for nonlinear ordinary and partial differential equations as shown in Material and Methods Section. Efficiency and effectiveness of the residual power series scheme in processing the variable-depth shallow water equations are obtained in Results part of this paper. High accurate semi-analytic solutions, with easily computable components, are derive and shown graphically. Absolute errors are computed and plotted on the whole domain. Numerical representation for the flow of water is also shown. In comparison to other existing methods, the presented Algorithm is simple and reliable which can be expanded to tackle a wide range of nonlinear evolution equations arise in physics and engineering.

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